DAMTP/96–99
FTUV/96–75 IFIC/96–84
hep-th/9612186
(to appear in J. Math. Phys.)

The Z_2-graded Schouten-Nijenhuis bracket and generalized super-Poisson structures *

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Abstract

The super or Z_2-graded Schouten-Nijenhuis bracket is introduced. Using it, new generalized super-Poisson structures are found which are given in terms of certain graded-skew-symmetric contravariant tensors Λ of even order. The corresponding super 'Jacobi identities' are expressed by stating that these ten-

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*Running Title: Super-Schouten bracket and super-Poisson structures.

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‡On sabbatical (J.A.) leave and on leave of absence (J.C.P.B.) from 2 above.

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sors have zero super Schouten-Nijenhuis bracket with themselves $[\Lambda, \Lambda] = 0$. As a particular case, we provide the linear generalized super-Poisson structures which can be constructed on the dual spaces of simple superalgebras with a non-degenerate Killing metric. The $su(3,1)$ superalgebra is given as a representative example.

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I. INTRODUCTION

We devote this paper to the introduction of the Z_2-graded (or 'super') Schouten-Nijenhuis bracket and to its application in the definition of super-Poisson brackets and structures, old and new, extending the approach of[1] to the Z_2-graded case. The generalization of the standard Poisson Brackets (PB) and Poisson structures (PS) proposed in[1] is different from that originally given by Nambu[2] some twenty years ago, later also considered in[3,4] and further extended by Takhtajan[5] (see[6] for a comparison). It is based on the consideration of the Schouten-Nijenhuis bracket (SNB)[7,8] which for the standard PS expresses[9,10] the Jacobi condition by requiring zero SNB, $[\Lambda, \Lambda] = 0$, for the (skew-symmetric) bivector field Λ defining the PS on the manifold M. The generalized Poisson structures (GPS) in[1] are then defined by skew-symmetric contravariant tensors of *even* order (even multivectors) $\Lambda^{(2p)}$. In this way, the skew-symmetry of the generalized Poisson bracket (which involves $2p$ functions) and the Leibniz rule are automatically incorporated by the properties of $\Lambda^{(2p)} \in \wedge^{2p}(M)$ (*odd* GPS have been introduced in[6]). The generalized Jacobi identity (GJI) is now geometrically expressed as $[\Lambda^{(2p)}, \Lambda^{(2p)}] = 0$, and is different from Takhtajan's 'fundamental identity' which expresses the fact that the time derivative (the 'adjoint' map) is a derivation of the n-bracket[5].

Graded Poisson structures have been considered before[11,12] (see also[13,14] and references therein). In[15] they were called supercanonical structures S, where the two-vector S was defined as having a vanishing graded Schouten bracket with itself. Poisson supermanifolds (see, *e.g.*[16] and references therein) require the replacement of the differentiable manifold M by a supermanifold Σ. By 'supermanifold' we understand here a finite dimensional topological space which has locally the structure of a *superspace i.e.*, a space the coordinates of which are given by the even and odd elements of a Grassmann algebra. We thus follow the 'geometric' (see[17,18]) rather that the 'algebraic' (see[19,20]) approach (see[21] for a comparison). The algebra of functions $\mathcal{F}(\Sigma)$ on Σ, endowed with a suitable Poisson bracket, becomes a Poisson superalgebra (see, *e.g.*,[22]). In order to generalize the (Z_2-graded or) super-Poisson

structures (SPS), it is convenient to introduce them through a Z_2-graded skew-symmetric contravariant tensor field of order two, or *superbivector*, and to express the super-Jacobi identity as the vanishing of a previously defined Z_2-graded SNB for Z_2-graded multivectors or *supermultivectors*. There are various types of algebras and brackets related to the original Schouten[7] and Nijenhuis constructions for multivector fields and differential forms[8,23] (see[24] and references therein for a discussion of various algebras). Here we shall consider only the mentioned case of the SNB for supermultivectors, and will not discuss other graded constructions as *e.g.*, those using vector-valued forms[25]. Thus, we shall start by introducing in Sec. II the super SNB for supermultivector fields.

Using the result of Sec. II, an outline of super-Poisson structures is presented in Sec. III from the Z_2-graded SNB point of view. Linear super-Poisson structures are introduced at the end of Sec. III; they are defined, as their standard (bosonic) counterparts, by the structure constants defining the corresponding Lie superalgebra. Generalized super-Poisson structures (GSPS) are defined in Sec. IV, and then the linear case is considered in Sec. V. In the standard bosonic case it is not difficult[1] to provide (an infinite number of) examples of (linear) GPS using the cohomology properties[26] (see also *e.g.*,[27]) of simple Lie algebras, and in fact these properties may be used to classify the linear GPS which may be constructed on them. Simple superalgebras, however, may have[28] a vanishing Killing form. In the nondegenerate case, nevertheless, the arguments are similar to those of the standard case, and turn out to be related to cohomology of Lie superalgebras. This is discussed in Sec. V; an example, that of $su(3,1)$, is given in Sec. VI, although its presentation makes it directly applicable to other superalgebras.

II. THE Z_2-GRADED SCHOUTEN-NIJENHUIS BRACKET

Before giving the local expression for the Z_2-graded SNB, it is convenient to establish the conventions. Let $\{x^i\}$ be local coordinates on a supermanifold Σ and let $\alpha(i) \equiv \alpha(x^i)$ be the Z_2-grade or Grassmann *parity* [0 (even or 'Bose') or 1 (odd or 'Fermi')] of x^i. Let

$\partial_i \equiv \partial/\partial x^i$ and dx^i be derivatives and one-forms. Then,

$$\partial_i dx^j = \delta_i^j \ , \ df = dx^i \partial_i f \ , \ (X_1 \otimes ... \otimes X_p)(\omega^1, ..., \omega^p) = (-1)^{\Delta_p(\omega, X)} X_1(\omega^1)...X_p(\omega^p) \ , \tag{II.1}$$

where δ_i^j is the usual Kronecker symbol, X_i and ω^j are one-vectors and one-forms respectively, and[18]

$$\Delta_p(\omega, X) = \sum_{\substack{r,s=1 \\ r<s}}^{p} \alpha(\omega^r)\alpha(X_s) \quad . \tag{II.2}$$

where $\alpha(\omega), \alpha(X)$ are the grades of ω and X respectively. For the wedge product we shall use

$$\partial_i \wedge \partial_j \equiv \partial_i \otimes \partial_j - (-1)^{\alpha(i)\alpha(j)} \partial_j \otimes \partial_i \tag{II.3}$$

(similarly for ω^i). Eq. (II.3) may be expressed as $\partial_i \wedge \partial_j = \tilde{\epsilon}_{ij}^{kl} \partial_k \otimes \partial_l$, where $\tilde{\epsilon}_{ij}^{kl} = -(-1)^{\alpha(k)\alpha(l)} \tilde{\epsilon}_{ij}^{lk}$.

The general 'exterior' product may be defined if we now introduce the tilded (Z_2-graded) Levi-Civita symbol $\tilde{\epsilon}_{j_1 \ldots j_p}^{i_1 \ldots i_p}$. We define $\tilde{\epsilon}$ by

$$\tilde{\epsilon}_{j_1 \ldots j_p}^{i_1 \ldots i_p} = \sum_{s=1}^{p} (-1)^{s-1} (-1)^{\alpha(i_s)[\alpha(i_1)+\ldots+\alpha(i_{s-1})]} \delta_{j_1}^{i_s} \tilde{\epsilon}_{j_2 \ \ldots \ j_p}^{i_1 \ldots \widehat{i_s} \ldots i_p} \quad , \quad \tilde{\epsilon}_j^i \equiv \delta_j^i \tag{II.4}$$

or, equivalently, by

$$\tilde{\epsilon}_{j_1 \ldots j_p}^{i_1 \ldots i_p} = \sum_{s=1}^{p} (-1)^{s-1} (-1)^{\alpha(j_s)[\alpha(j_1)+\ldots+\alpha(j_{s-1})]} \delta_{j_s}^{i_1} \tilde{\epsilon}_{j_1 \ldots \widehat{j_s} \ldots j_p}^{i_2 \ \ldots \ i_p} \quad . \tag{II.5}$$

Clearly, $\tilde{\epsilon}$ reduces to the standard ϵ in the Bose case ($\alpha(i) = 0$) for which

$$\epsilon_{j_1 \ldots j_p}^{i_1 \ldots i_p} = \begin{vmatrix} \delta_{j_1}^{i_1} & \cdots & \delta_{j_p}^{i_1} \\ \vdots & & \vdots \\ \delta_{j_1}^{i_p} & \cdots & \delta_{j_p}^{i_p} \end{vmatrix} \quad . \tag{II.6}$$

Expressions (II.4) and (II.5) correspond, respectively, to expanding this determinant by columns/rows in such a way that the column (j)/row (i) indices are written in natural order, and then adding the sign factors which would be needed to bring the row (i)/ column

(j) indices to the ordering in which they actually appear. The Z_2-graded $\tilde{\epsilon}$ has the graded antisymmetry property

$$\tilde{\epsilon}^{i_1 \, \ldots \, i_p}_{j_1 \ldots j_k j_{k+1} \ldots j_p} = -(-1)^{\alpha(j_k)\alpha(j_{k+1})} \tilde{\epsilon}^{i_1 \, \ldots \, i_p}_{j_1 \ldots j_{k+1} j_k \ldots j_p} \tag{II.7}$$

and similarly for superscripts. As in the even case, we have

$$\tilde{\epsilon}^{i_1 \ldots i_{p-1} l_p \ldots l_n}_{j_1 \quad \ldots \quad j_n} \tilde{\epsilon}^{i_p \ldots i_n}_{l_p \ldots l_n} = (n-p+1)! \, \tilde{\epsilon}^{i_1 \ldots i_n}_{j_1 \ldots j_n} \quad . \tag{II.8}$$

Using (II.7) we may now write

$$\partial_{i_1} \wedge \ldots \wedge \partial_{i_p} := \tilde{\epsilon}^{j_1 \ldots j_p}_{i_1 \ldots i_p} \partial_{j_1} \otimes \ldots \otimes \partial_{j_p} \quad . \tag{II.9}$$

Then, the exterior product of two general supermultivectors A and B of parities $\alpha(A)$, $\alpha(B)$ and of order p and q, respectively, locally expressed by

$$A = \frac{1}{p!} A^{i_1 \ldots i_p} \partial_{i_1} \wedge \ldots \wedge \partial_{i_p} \quad , \quad B = \frac{1}{q!} B^{j_1 \ldots j_q} \partial_{j_1} \wedge \ldots \wedge \partial_{j_q} \quad , \tag{II.10}$$

is given by

$$\begin{aligned} A \wedge B \equiv & \frac{1}{p!q!} (-1)^{[\alpha(i_1)+\ldots+\alpha(i_p)][\alpha(j_1)+\ldots+\alpha(j_q)+\alpha(B)]} \cdot \\ & \cdot A^{i_1 \ldots i_p} B^{j_1 \ldots j_q} \partial_{i_1} \wedge \ldots \wedge \partial_{i_p} \wedge \partial_{j_1} \wedge \ldots \wedge \partial_{j_q} \\ = & \frac{1}{(p+q)!} (A \wedge B)^{i_1 \ldots i_p j_1 \ldots j_q} \partial_{i_1} \wedge \ldots \wedge \partial_{i_p} \wedge \partial_{j_1} \wedge \ldots \wedge \partial_{j_q} \quad , \end{aligned} \tag{II.11}$$

where

$$(A \wedge B)^{i_1 \ldots i_p j_1 \ldots j_q} = \frac{1}{p!q!} (-1)^{[\alpha(k_1)+\ldots+\alpha(k_p)][\alpha(l_1)+\ldots+\alpha(l_q)+\alpha(B)]} \tilde{\epsilon}^{i_1 \ldots i_p j_1 \ldots j_q}_{k_1 \ldots k_p l_1 \ldots l_q} A^{k_1 \ldots k_p} B^{l_1 \ldots l_q} \quad . \tag{II.12}$$

The contravariant exterior algebra of supermultivectors on Σ will be denoted by $\wedge(\Sigma)$; a vector field X belongs to $\wedge^1(\Sigma)$ and $\mathcal{F}(\Sigma) = \wedge^0(\Sigma)$. Due to the Z_2-gradation (see (II.3)), there may exist supermultivectors of arbitrary order p irrespective of the dimension of Σ if Σ is not an ordinary manifold.

We are now in position to generalize the Schouten-Nijenhuis bracket to the supersymmetric case. The way to proceed is the following. On an ordinary manifold M it is known

that the SNB of $X_1\wedge\ldots\wedge X_p$ and $Y_1\wedge\ldots\wedge Y_q$, $X_i, Y_j\in\wedge^1(M)$, is the bilinear local operation given by

$$\begin{aligned} &[X_1\wedge...\wedge X_p\ ,\ Y_1\wedge\ldots\wedge Y_q]= \\ &=\sum_{s=1,t=1}^{p,q}(-1)^{t+s}X_1\wedge...\hat{X}_s...\wedge X_p\wedge[X_s,Y_t]\wedge Y_1\wedge...\hat{Y}_t...\wedge Y_q \quad . \end{aligned}\tag{II.13}$$

This formula is the result of extending the Lie derivative of vector fields on M, $L_X Y=[X,Y]$, in a natural (and unique) way to arbitrary elements of the contravariant exterior algebra $\wedge(M)$. Now, substituting Σ for M, it is not difficult to see that for $X_i, Y_j\in\wedge^1(\Sigma)$ (II.13) is replaced by

$$\begin{aligned} &[X_1\wedge...\wedge X_p\ ,\ Y_1\wedge\ldots\wedge Y_q]= \\ &=\sum_{s=1,t=1}^{p,q}(-1)^{t+s+\alpha(Y_t)[\alpha(Y_1)+...+\alpha(Y_{t-1})]+\alpha(X_s)[\alpha(X_{s+1})+...+\alpha(X_p)]} \\ &X_1\wedge...\hat{X}_s...\wedge X_p\wedge[X_s,Y_t]\wedge Y_1\wedge...\hat{Y}_t...\wedge Y_q \end{aligned}\tag{II.14}$$

where in the $r.h.s$ $[\ ,\ ]$ denotes the Z_2-graded SNB of two vector fields (which is an anti-commutator if both are odd). Using (II.14) we may now introduce the SSNB for arbitrary elements of $\wedge(\Sigma)$. This leads us to the following local definition:

Definition II.1 (*Super Schouten-Nijenhuis bracket*)

Let $A\in\wedge^p(\Sigma)$ and $B\in\wedge^q(\Sigma)$ be two supermultivectors on a supermanifold Σ given locally by (II.10). The super Schouten-Nijenhuis bracket (SSNB) $[A,B]$ is the (super-)bilinear operation of local type $[\ ,\ ]:\wedge^p(\Sigma)\times\wedge^q(\Sigma)\to\wedge^{p+q-1}(\Sigma)$ locally defined by

$$\begin{aligned} [A,B]=&\frac{1}{(p-1)!q!}(-1)^{[\alpha(i_1)+...+\alpha(i_{p-1})][\alpha(j_1)+...+\alpha(j_q)+\alpha(B)]}A^{\nu i_1\ldots i_{p-1}} \\ &\partial_\nu B^{j_1\ldots j_q}\partial_{i_1}\wedge...\wedge\partial_{i_{p-1}}\wedge\partial_{j_1}\wedge...\wedge\partial_{j_q} \\ &+\frac{(-1)^p}{p!(q-1)!}(-1)^{\alpha(A)[\alpha(j_1)+...+\alpha(j_{q-1})+\alpha(B)]}B^{\nu j_1\ldots j_{q-1}} \\ &\partial_\nu A^{i_1\ldots i_p}\partial_{i_1}\wedge...\wedge\partial_{i_p}\wedge\partial_{j_1}\wedge...\wedge\partial_{j_{q-1}} \quad , \end{aligned}\tag{II.15}$$

where $\alpha(i_k)$, $\alpha(j_k)$ ($\alpha(A)$, $\alpha(B)$) are the Grassmann parities of the corresponding coordinates (supermultivectors A and B); $\alpha([A,B])=\alpha(A)+\alpha(B)$.

If we write now $[A,B]=\frac{1}{(p+q-1)!}[A,B]^{i_1\ldots i_{p+q-1}}\partial_{i_1}\wedge\ldots\wedge\partial_{i_{p+q-1}}$, the coordinates of the SSNB are given by [eq. (II.12)]

$$\begin{aligned}
[A,B]^{k_1\ldots k_{p+q-1}} = &\frac{1}{(p-1)!q!}(-1)^{[\alpha(i_1)+\ldots+\alpha(i_{p-1})][\alpha(j_1)+\ldots+\alpha(j_q)+\alpha(B)]} \\
&\tilde{\epsilon}^{k_1\ldots k_{p+q-1}}_{i_1\ldots i_{p-1}j_1\ldots j_q}A^{\nu i_1\ldots i_{p-1}}\partial_\nu B^{j_1\ldots j_q} \\
&+\frac{(-1)^p}{p!(q-1)!}(-1)^{\alpha(A)[\alpha(j_1)+\ldots+\alpha(j_{q-1})+\alpha(B)]} \\
&\tilde{\epsilon}^{k_1\ldots k_{p+q-1}}_{i_1\ldots i_p j_1\ldots j_{q-1}}B^{\nu j_1\ldots j_{q-1}}\partial_\nu A^{i_1\ldots i_p} \quad .
\end{aligned} \tag{II.16}$$

Expression (II.15) reproduces the definition of the graded commutator when A and B are graded vector fields on Σ ($A,B\in\wedge^1(\Sigma)$). It also reduces to the local expression of the usual Schouten-Nijenhuis bracket (see, *e.g.*[9]) for the bosonic case ($\alpha(x^i)=0,\ \alpha(A)=0,\ \alpha(B)=0$) as it should. It follows from (II.15) that the SSNB has the following property ($A\in\wedge^p(\Sigma), B\in\wedge^q(\Sigma)$):

$$[A,B]=(-1)^{pq}(-1)^{\alpha(A)\alpha(B)}[B,A] \quad . \tag{II.17}$$

As a result, $[A,A]$ is identically zero for a Grassmann even p-multivector if p is *odd*, and $[A,A]=0$ is a non-trivial equation if A is of zero parity and p is even. Also, if $C\in\wedge^r(\Sigma)$

$$\begin{aligned}
&(-1)^{\alpha(A)\alpha(C)}(-1)^{pr}[[A,B],C]+(-1)^{\alpha(B)\alpha(A)}(-1)^{qp}[[B,C],A] \\
&\quad +(-1)^{\alpha(C)\alpha(B)}(-1)^{rq}[[C,A],B]=0 \quad ,
\end{aligned} \tag{II.18}$$

$$\begin{aligned}
&[A,B\wedge C]=[A,B]\wedge C+(-1)^{(p-1)q}(-1)^{\alpha(B)\alpha(A)}B\wedge[A,C] \quad , \\
&[A\wedge B,C]=(-1)^pA\wedge[B,C]+(-1)^{rq}(-1)^{\alpha(B)\alpha(C)}[A,C]\wedge B \quad .
\end{aligned} \tag{II.19}$$

Notice that if A is a vector field, the first in (II.19) is just the derivation property

$$L_A(B\wedge C)=(L_AB)\wedge C+(-1)^{\alpha(A)\alpha(B)}B\wedge(L_AC) \quad , \tag{II.20}$$

where L_A is the Lie derivative with respect to A.

The dependence on p,q of eqs. (II.17) and (II.18) indicates that the definition of the SSNB in eq. (II.15) does not have the usual properties of a superalgebra bracket with respect

to the Grassmann parity grading. This may be achieved if (II.15) is slightly modified and a new grading π for supermultivectors A is introduced. The *degree* $\pi(A) \equiv a$ is defined by

$$\pi(A) := \alpha(A) + p - 1 \quad , \tag{II.21}$$

where p is the order of the supermultivector A. It then follows that $\pi([A,B]) = \alpha(A) + \alpha(B) + (p+q-1) - 1 = \pi(A) + \pi(B) \equiv a + b$. If we now define a new, primed SSNB by

$$[A,B]' = (-1)^{p+1}(-1)^{\alpha(A)(q+1)}[A,B] \quad , \tag{II.22}$$

where $[A,B]$ is the old one given by (II.15), properties (II.17) and (II.18) now adopt the superalgebra form (see, *e.g.*,[29,18])

$$[A,B]' = -(-1)^{ab}[B,A]' \quad , \tag{II.23}$$

$$(-1)^{ac}[[A,B]',C]' + (-1)^{cb}[[C,A]',B]' + (-1)^{ba}[[B,C]',A]' = 0 \quad , \tag{II.24}$$

where eq. (II.24) is the standard super-Jacobi identity; $[A,A]$ is trivially zero for a even. Thus, the primed SSNB bracket above extends the superalgebra of supervector fields X (for which the parity associated with π is just the Grassmann one $\pi(X) = \alpha(X)$, eq. (II.21)), and makes a superalgebra of the exterior algebra of supermultivectors endowed with the SSNB.

III. SUPER-POISSON STRUCTURES

Let $\Sigma = \Sigma_0 \oplus \Sigma_1$ be a supermanifold and $\mathcal{F}(\Sigma) = \mathcal{F}_0(\Sigma) \oplus \mathcal{F}_1(\Sigma)$ be the algebra of Z_2-graded smooth functions on Σ; $f \in \mathcal{F}_0(\Sigma)$ $[\mathcal{F}_1(\Sigma)]$ is said to be homogeneous of even [odd] parity.

Definition III.1 (*Super-Poisson bracket*)

A *super-Poisson bracket* $\{\cdot,\cdot\}$ (SPB) on $\mathcal{F}(\Sigma)$ is a bilinear operation assigning to every pair of functions $f, g \in \mathcal{F}(\Sigma)$ a new function $\{f,g\} \in \mathcal{F}(\Sigma)$, such that for homogeneous functions satisfies the following conditions:

a) grade zero super-Poisson bracket

$$\alpha(\{f,g\}) \equiv \alpha(f)+\alpha(g)\ (\mathrm{mod}\ 2) \quad ; \tag{III.1}$$

b) super skew-symmetry

$$\{f,g\} = -(-1)^{\alpha(f)\alpha(g)}\{g,f\}\ ; \tag{III.2}$$

c) graded Leibniz rule (derivation property)

$$\{f,gh\} = \{f,g\}h + (-1)^{\alpha(f)\alpha(g)}g\{f,h\} = \{f,g\}h + (-1)^{\alpha(g)\alpha(h)}\{f,h\}g \quad , \tag{III.3}$$

d) super-Jacobi identity

$$\begin{aligned}\frac{1}{2}\mathrm{sAlt}\{f_1,\{f_2,f_3\}\} \equiv & (-1)^{\alpha(1)\alpha(3)}\{f_1,\{f_2,f_3\}\} + (-1)^{\alpha(2)\alpha(1)}\{f_2,\{f_3,f_1\}\} \\ & + (-1)^{\alpha(3)\alpha(2)}\{f_3,\{f_1,f_2\}\} = 0\ ,\end{aligned} \tag{III.4}$$

where $\alpha(i) \equiv \alpha(f_i)$, $i=1,2,3$ and sAlt means 'super' or Z_2-graded alternation. Since the identities (III.2), (III.4) are just the axioms of a superalgebra, the space $\mathcal{F}(\Sigma)$ endowed with the SPB $\{\cdot,\cdot\}$ becomes an (infinite-dimensional) superalgebra, and Σ is a *super-Poisson space*. The first of the above conditions means that the SPB operation $\{\cdot,\cdot\}$ itself is Grassmann even. We shall restrict ourselves here to this case although odd PB ('antibrackets') for which $\alpha(\{f,g\}) = \alpha(f)+\alpha(g)+1$ appear in the theory of odd supermechanics[11] (see also[12] and the Remark below).

Remark. Note the way the grading $[\alpha]$ has been defined. Odd structures have also appeared in mathematics in connection with the SNB (see[30]) and in physics, as in the Batalin-Vilkovisky formalism[31–35]; see also[24] and references therein and in[36].

Let x^j be coordinates on Σ and consider SPB of the form

$$\{f(x),g(x)\} := (-1)^{\alpha(f)\alpha(k)+\alpha(j)\alpha(k)}\omega^{jk}(x)\partial_j f\partial_k g \quad , \quad j,k=1,\ldots,\mathrm{dim}\Sigma \quad , \tag{III.5}$$

where $\alpha(i)$ is as before and $\partial_j = \partial/\partial x^j$ is a left derivative. Clearly, it satisfies (III.2). Note that, in particular,

$$\{x^i, x^j\} = \omega^{ij} \tag{III.6}$$

and that we take here $\alpha(\omega^{ij}) = \alpha(x^i) + \alpha(x^j)$. Since the graded Leibniz rule is automatically guaranteed by (III.5), $\omega^{ij}(x)$ defines a SPB if $\omega^{ij}(x) = -(-1)^{\alpha(i)\alpha(j)}\omega^{ji}(x)$ and eq. (III.4) is satisfied *i.e.*, if

$$(-1)^{\alpha(j)\alpha(m)}\omega^{jk}\partial_k\omega^{lm} + (-1)^{\alpha(l)\alpha(j)}\omega^{lk}\partial_k\omega^{mj} + (-1)^{\alpha(m)\alpha(l)}\omega^{mk}\partial_k\omega^{jl} = 0 \ , \tag{III.7}$$

which is equivalent (cf. (III.4)) to $s\mathrm{Alt}[\omega^{jk}\partial_k\omega^{lm}] := \frac{1}{2}\tilde{\epsilon}^{j\,l\,m}_{i_1 i_2 i_3}\omega^{i_1 k}\partial_k\omega^{i_2 i_3} = 0$.

The requirements (III.1), (III.2) and (III.3) imply that the SPB may be given in terms of a Z_2-graded bivector field or *super-Poisson bivector* $\Lambda \in \wedge^2(\Sigma)$ of zero parity. Locally,

$$\Lambda = \frac{1}{2}(-1)^{\alpha(j)\alpha(k)}\omega^{jk}\partial_j \wedge \partial_k = -\frac{1}{2}\omega^{kj}\partial_j \wedge \partial_k \quad . \tag{III.8}$$

Condition (III.7) may now be expressed in terms of Λ and the SSNB (eq. (II.15)) as $[\Lambda, \Lambda] = 0$. Thus, if x^j, x^k are both odd, $\alpha(x^j) = \alpha(x^k) = 1$, ω^{jk} is symmetric rather than antisymmetric. A super bivector $\Lambda \in \wedge^2(\Sigma)$ such that $[\Lambda, \Lambda] = 0$ defines a *super-Poisson structure* on Σ and Σ itself becomes a *super-Poisson space*. The SPB is then given by

$$\{f, g\} = \Lambda(df, dg) \quad , \quad f, g \in \mathcal{F}(\Sigma) \quad . \tag{III.9}$$

Two SPS Λ_1, Λ_2 on Σ are *compatible* if any linear combination of them is again a SPS. In terms of the SSNB this means that $[\Lambda_1, \Lambda_2] = 0$.

Given a bosonic function $H \in \mathcal{F}_0(\Sigma)$, the supervector field $X_H = i_{dH}\Lambda$ (where $i_\alpha\Lambda(\beta) := \Lambda(\alpha, \beta)$, α, β one-forms), is called a *super-Hamiltonian vector field* of H. From the super-Jacobi identity (III.4) easily follows that

$$[X_f, X_H] = X_{\{f,H\}} \quad . \tag{III.10}$$

Thus, the super Hamiltonian vector fields span a sub-superalgebra of the superalgebra $\mathcal{X}(\Sigma)$ of all smooth supervector fields on Σ. In local coordinates

$$X_H = (-1)^{\alpha(j)\alpha(k)}\omega^{jk}(x)\partial_j H \partial_k \quad ; \quad X_H.f = \{H, f\} \ . \tag{III.11}$$

A particular case is that of the *linear super-Poisson structures*. A real finite–dimensional superalgebra $\mathcal{G}$ with Z_2-graded Lie bracket $[\,.,.]$ defines in a natural way a SPB $\{\,.,.\}_{\mathcal{G}}$ on the dual space $\mathcal{G}^*$ of $\mathcal{G}$. The natural identification $\mathcal{G} \cong (\mathcal{G}^*)^*$, allows us to think of $\mathcal{G}$ as a subset of the ring of smooth functions $\mathcal{F}(\mathcal{G}^*)$. Choosing a linear basis $\{\,e_i\,\}_{i=1}^r$ of $\mathcal{G}$, and identifying its components with linear coordinate functions x_i on the dual space $\mathcal{G}^*$ by means of $x_i(x) = \langle x, e_i\rangle$ for all $x \in \mathcal{G}^*$, the fundamental SPB on $\mathcal{G}^*$ may be defined by

$$\{x_i, x_j\}_{\mathcal{G}} = x_k C_{ij}^k \quad , \quad i,j,k = 1,\ldots,r = \dim\mathcal{G} \quad , \tag{III.12}$$

using that $[e_i, e_j] = C_{ij}^k e_k$, where C_{ij}^k are the structure constants of $\mathcal{G}$. Since these are of even Grassmann parity, assumption a) in Def. III.1 tells us that $\alpha(\{x_i, x_j\}) = \alpha(x_k)$ as is indeed the case. Intrinsically, the SPB $\{.,.\}_{\mathcal{G}}$ on $\mathcal{F}(\mathcal{G}^*)$ is defined by

$$\{f,g\}_{\mathcal{G}}(x) = \langle x, [df(x), dg(x)]\rangle \quad , \quad f,g \in \mathcal{F}(\mathcal{G}^*), x \in \mathcal{G}^* \quad ; \tag{III.13}$$

locally, $[df(x), dg(x)] = (-1)^{\alpha(f)\alpha(j)+\alpha(i)\alpha(j)} e_k C_{ij}^k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$, $\{f,g\}_{\mathcal{G}}(x) = (-1)^{\alpha(i)\alpha(j)+\alpha(f)\alpha(j)}$ $x_k C_{ij}^k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$. The above SPB $\{.,.\}_{\mathcal{G}}$ will be called a *super Lie–Poisson bracket*. It is associated to a two-supervector field $\Lambda_{\mathcal{G}}$ on $\mathcal{G}^*$ locally defined by

$$\Lambda_{\mathcal{G}} = (-1)^{\alpha(i)\alpha(j)}\frac{1}{2} x_k C_{ij}^k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \equiv -\frac{1}{2}\omega_{ji}\partial^i \wedge \partial^j \tag{III.14}$$

(cf. (III.8)), so that (cf. (III.9)) $\Lambda_{\mathcal{G}}(df, dg) = \{f,g\}_{\mathcal{G}}$. Since $\alpha(x_i) = \alpha(\partial^i)$, we see that condition a) in Def III.1 implies that $\alpha(\Lambda_{\mathcal{G}}) = 0$ and that accordingly the degree $\pi(\Lambda_{\mathcal{G}})$ of $\Lambda_{\mathcal{G}}$ reduces to $(\mathrm{order}(\Lambda_{\mathcal{G}}) - 1) = 1$. We conclude by noting that the non-trivial condition $[\Lambda_{\mathcal{G}}, \Lambda_{\mathcal{G}}] = 0$ (cf. (III.7)) reproduces the super-Jacobi identity for the superalgebra $\mathcal{G}$ written as

$$\frac{1}{2}\mathrm{sAlt}(C_{i_1\rho}^{\sigma} C_{i_2 i_3}^{\rho}) := \frac{1}{2}\tilde{\epsilon}_{i_1 i_2 i_3}^{j_1 j_2 j_3} C_{j_1\rho}^{\sigma} C_{j_2 j_3}^{\rho} = 0 \quad . \tag{III.15}$$

IV. GENERALIZED SUPER-POISSON STRUCTURES

A rather stringent condition needed to define a SPS on a supermanifold is the super-Jacobi identity (III.4); it will be fulfilled if the coordinates of the super-Poisson bivector

(III.8) satisfy (III.7). This is expressed in a geometrical way by the vanishing of the SSNB of $\Lambda \equiv \Lambda^{(2)}$ with itself, $[\Lambda^{(2)}, \Lambda^{(2)}] = 0$. So, it seems natural to consider generalizations of the SPS in terms of $2p$-ary operations determined by Grassmann even $2p$-supermultivector fields $\Lambda^{(2p)}$, the case $p = 1$ being the standard one. Note that, if we relax condition a) in Def. III.1, we may have odd Poisson brackets and structures, defined in this case by Grassmann odd q-supermultivectors also of odd order (and hence also of odd π-parity, eq. (II.21)) for which $[\Lambda, \Lambda] = 0$ (cf. (II.17)) will be non-trivial.

Having this in mind, let us introduce first

Definition IV.1 (*Generalized super-Poisson bracket*)
A generalized super-Poisson bracket (GSPB) $\{\cdot, \cdot, \ldots, \cdot, \cdot\}$ on a supermanifold Σ is a mapping $\mathcal{F}(\Sigma) \times \overset{2p}{\cdots} \times \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma)$ assigning a function $\{f_1, f_2, \ldots, f_{2p}\}$ to every set $f_1, \ldots, f_{2p} \in \mathcal{F}(\Sigma)$ which is linear in all arguments and satisfies the following conditions:

a) even GSPB

$$\alpha(\{f_1, f_2, \ldots, f_{2p}\}) \equiv \alpha(f_1) + \ldots + \alpha(f_{2p}) \pmod 2 \quad , \qquad \text{(IV.1)}$$

b) graded skew-symmetry in all f_j;

c) graded Leibniz rule: $\forall f_i, g, h \in \mathcal{F}(\Sigma)$,

$$\{f_1, \ldots, f_{2p-1}, gh\} = \{f_1, \ldots, f_{2p-1}, g\}h + (-1)^{\alpha(g)\alpha(h)}\{f_1, \ldots, f_{2p-1}, h\}g \quad ; \qquad \text{(IV.2)}$$

d) generalized super-Jacobi identity: $\forall f_i \in \mathcal{F}(\Sigma)$,

$$\mathrm{sAlt}\,\{f_1, f_2, \ldots, f_{2p-1}, \{f_{2p}, \ldots, f_{4p-1}\}\} = 0 = \tilde{\epsilon}^{j_1 \ldots j_{4p-1}}_{1 \ldots 4p-1}\{f_{j_1}, f_{j_2}, \ldots, f_{j_{2p-1}}, \{f_{j_{2p}}, \ldots, f_{j_{4p-1}}\}\} \,. \qquad \text{(IV.3)}$$

The property a) indicates that we are again restricting ourselves to a Grassmann even GSPB. Conditions b) and c) imply that the GSPB is given by a super skew-symmetric multiderivative, *i.e.* by a Grassmann even $2p$-supermultivector field $\Lambda^{(2p)} \in \wedge^{2p}(\Sigma)$. Condition (IV.3) will be called the *generalized super-Jacobi identity*; for $p = 2$ it contains 35 terms

(C^{2p-1}_{4p-1} in the general case). It may be rewritten as $[\Lambda^{(2p)}, \Lambda^{(2p)}] = 0$ where $\Lambda^{(2p)}$ defines a generalized SPS on Σ. In the bosonic case, where α is always zero and sAlt reduces to Alt, these generalized Poisson structures have been proposed in[1]. Clearly, the above conditions reproduce (III.1)–(III.4) for $p = 1$. The compatibility condition in Sec. III for $p = 1$ may be now extended in the following sense: two generalized super-Poisson structures $\Lambda^{(2p)}$ and $\Lambda^{(2q)}$ on Σ are called *compatible* if they 'commute' under the SSNB *i.e.*, $[\Lambda^{(2p)}, \Lambda^{(2q)}] = 0$. For the linear case, our generalized SPS are automatically obtained from constant supermultivectors of order $2p + 1$.

Let x^j be local coordinates on $U \subset \Sigma$. Then the GSPB has the form

$$\{f_1(x), f_2(x), \ldots, f_{2p}(x)\} = (-1)^{\Delta_{2p}(f,j)}(-1)^{\Delta_{2p}(j,j)}\omega_{j_1 j_2 \cdots j_{2p}}\partial^{j_1} f_1\, \partial^{j_2} f_2\, \ldots\, \partial^{j_{2p}} f_{2p} \quad , \tag{IV.4}$$

where $\Delta_{2p}(f, j) = \sum_{r<s}^{2p} \alpha(f_r)\alpha(j_s)$, $\Delta_{2p}(j, j) = \sum_{r<s}^{2p} \alpha(j_r)\alpha(j_s)$ (cf. (II.2)) and $\omega_{j_1 j_2 \cdots j_{2p}}$ are the coordinates of a graded skew-symmetric tensor which satisfies

$$\mathrm{sAlt}\,(\omega_{j_1 j_2 \ldots j_{2p-1} k}\, \partial^k\, \omega_{j_{2p} \ldots j_{4p-1}}) = 0 \tag{IV.5}$$

as a result of (IV.3) (notice that this would be false for a bracket with an odd number of arguments). In terms of an even supermultivectior field of order $2p$ the generalized super-Poisson structure is defined by the $2p$-vector

$$\Lambda^{(2p)} = \frac{1}{(2p)!}(-1)^{\Delta_{2p}(j,j)}\omega_{j_1 \ldots j_{2p}}\, \partial^{j_1} \wedge \ldots \wedge \partial^{j_{2p}} = \frac{(-1)^p}{(2p)!}\omega_{j_{2p} \ldots j_1}\partial^{j_1} \wedge \ldots \wedge \partial^{j_{2p}} \tag{IV.6}$$

and, using (II.1)

$$\Lambda(df_1, \ldots, df_{2p}) = \{f_1, \ldots, f_{2p}\} \quad . \tag{IV.7}$$

Lemma IV.1

The vanishing of the SSNB $[\Lambda^{(2p)}, \Lambda^{(2p)}] = 0$ reproduces eq. (IV.5).

Proof: Let $\Lambda^{(2p)}$ be the $2p$-vector defined in (IV.6). To show this, it suffices to use (II.15) for the case $A = B = \Lambda^{(2p)}$, $\alpha(\Lambda^{(2p)}) = 0$ since we assumed that the Grassmann parity of the GPB was determined by those of the $2p$ functions involved in it only. Then,

$$
\begin{aligned}
[\Lambda^{(2p)},\Lambda^{(2p)}] = -&\frac{1}{(2p-1)!(2p)!}(-1)^{[\alpha(i_1)+\ldots+\alpha(i_{2p-1})][\alpha(j_1)+\ldots+\alpha(j_{2p})]}\omega_{i_1\ldots i_{2p-1}\nu} \\
&(-1)^{\Delta_{2p-1}(i,i)+\Delta_{2p}(j,j)}\partial^\nu\omega_{j_1\ldots j_{2p}}\partial^{i_1}\wedge\ldots\wedge\partial^{i_{2p-1}}\wedge\partial^{j_1}\wedge\ldots\wedge\partial^{j_{2p}} \\
&-\frac{1}{(2p-1)!(2p)!}\omega_{j_1\ldots j_{2p-1}\nu}(-1)^{\Delta_{2p-1}(j,j)+\Delta_{2p}(i,i)} \\
&\partial^\nu\omega_{i_1\ldots i_{2p}}\partial^{i_1}\wedge\ldots\wedge\partial^{i_{2p}}\wedge\partial^{j_1}\wedge\ldots\wedge\partial^{j_{2p-1}} \\
&= -\frac{2}{(2p-1)!(2p)!}(-1)^{\Delta_{4p-1}(i,i)}\omega_{i_1\ldots i_{2p-1}\nu}\partial^\nu\omega_{i_{2p}\ldots i_{4p-1}}\partial^{i_1}\wedge\ldots\wedge\partial^{i_{4p-1}} \quad ,
\end{aligned}
\tag{IV.8}
$$

which, since $\Delta_{4p-1}(i,i)$ is invariant under reorderings of the indices $i_1,\ldots,i_{4p-1}$, gives condition (IV.5) if the SSNB is zero, *q.e.d.*

V. LINEAR GENERALIZED SUPER-POISSON STRUCTURES ON THE DUALS OF SIMPLE LIE SUPERALGEBRAS

Given a finite-dimensional Lie superalgebra $\mathcal{G}$, we know from Sec. III that there is a linear super-Poisson structure defined through the structure constants. If $\mathcal{G}$ admits a non-degenerate Killing metric k_{ij}, one may, on the other hand, construct the graded skew-symmetric order three tensor

$$
\omega(e_i,e_j,e_k) := k([e_i,e_j],e_k) = C_{ij}^l k_{lk} = C_{ijk}\ ,\ e_i\in\mathcal{G} \quad (i,j,k=1,\ldots,r=\dim\mathcal{G}) \tag{V.1}
$$

where the e_i are elements of a basis of $\mathcal{G}$. This tensor is invariant, *i.e.*

$$
\omega([e_l,e_i],e_j,e_k)+(-1)^{\alpha(l)\alpha(i)}\omega(e_i,[e_l,e_j],e_k)+(-1)^{[\alpha(i)+\alpha(j)]\alpha(l)}\omega(e_i,e_j,[e_l,e_k])=0 \quad . \tag{V.2}
$$

In general, the invariance (or *ad*-invariance) of a tensor of components $k_{i_1\ldots i_m}$ may be expressed as

$$
\sum_{s=1}^{m}(-1)^{\alpha(j)[\alpha(i_1)+\ldots+\alpha(i_{s-1})]}C_{ji_s}^{\rho}k_{i_1\ldots i_{s-1}\rho i_{s+1}\ldots i_m}=0 \quad , \tag{V.3}
$$

which for the case of a graded skew-symmetric tensor ω can be written as

$$
\tilde{\epsilon}^{j_1\ldots j_m}_{i_1\ldots i_m}C_{kj_1}^{\rho}\omega_{\rho j_2\ldots j_m}=0 \quad . \tag{V.4}
$$

Since we are assuming that k is non-degenerate, it can be used to raise indices as well, so starting from C_{ijk} as defined in (V.1), one can recover the structure constants and the corresponding super-Poisson structure. This fact was used in[1] to obtain linear generalized Poisson structures from simple Lie algebras by using certain invariant skew-symmetric forms of odd order (Lie algebra cohomology cocycles). These forms were obtained starting from ad-invariant symmetric polynomials (Casimirs), which are completely classified for simple Lie algebras. However, in contrast with this case, the ring of Casimir operators for simple superalgebras (the center $Z(U(\mathcal{G}))$ of the enveloping algebra) is not finitely generated in general (among the classical superalgebras this is the case only for $osp(1,2n)$ for which $Z(U(\mathcal{G}))$ is generated by n Casimir operators of order $2,4,\ldots,2n$). At the same time, the study of the invariant polynomials for superalgebras is much more involved than for the ordinary simple Lie algebras case (see in this respect[37–39,19,40] and references therein). Also, there is the problem that for simple superalgebras the Killing form may be zero since the invariance and simplicity entails that k is either non-degenerate or identically zero (k is non-degenerate for the following classical superalgebras: $A(m,n)\,,\ m>n\geq 0$ $[sl(m+1,n+1)]$; $B(m,n)\,,\ m\geq 0\,,\ n\geq 1$ $[osp(2m+1,2n)]$; $C(n)\,,\ n\geq 2$ $[osp(2,2n-2)]$; $D(m,n)\,,\ m\geq 2\,,\ n\geq 1\,,\ m\neq n+1$ $[osp(2m,2n)]$; $F(4)$ and $G(3)$[28]; see also[18]). We shall assume here that the Killing form is non-degenerate and consider Casimir operators defined by ad-invariant supersymmetric polynomials. We shall describe now how to obtain linear super-Poisson structures in this case.

Theorem V.1 (*Linear generalized SPS on a simple superalgebra*)

Let $\mathcal{G}$ be a simple superalgebra, and let $k_{i_1\ldots i_m}$ be a primitive non-trivial invariant graded-symmetric polynomial of order m. Then, the tensor $\omega_{\rho l_2\ldots l_{2m-2}\sigma}$

$$\omega_{\rho l_2\ldots l_{2m-2}\sigma} := \tilde{\epsilon}^{j_2\ldots j_{2m-2}}_{l_2\ldots l_{2m-2}}\tilde{\omega}_{\rho j_2\ldots j_{2m-2}\sigma}\ ,\ \tilde{\omega}_{\rho j_2\ldots j_{2m-2}\sigma} := k_{\rho i_1\ldots i_{m-1}} C^{i_1}_{j_2 j_3}\ldots C^{i_{m-1}}_{j_{2m-2}\sigma} \tag{V.5}$$

is completely graded skew-symmetric and

$$\Lambda^{(2m-2)} = \frac{(-1)^{m-1}}{(2m-2)!} x_\sigma \omega^{\sigma}_{\cdot l_1\ldots l_{2m-2}} \partial^{l_{2m-2}}\wedge\ldots\wedge\partial^{l_1} \tag{V.6}$$

defines a linear generalized super-Poisson structure on $\mathcal{G}$ by

$$\{x_{i_1}, \ldots, x_{i_{2m-2}}\} = x_\sigma \omega^{\sigma}_{\cdot i_1 \ldots i_{2m-2}} \quad . \tag{V.7}$$

Proof: Let us first consider the complete graded skew-symmetry. Since $\omega_{\rho l_2 \ldots l_{2m-2}\sigma}$ is, by (V.5), graded skew-symmetric under the interchange of l_i, l_j $i,j = 2, ..., 2m-2$ and under the interchange of σ and l_i, it suffices to prove the graded skew-symmetry relative to the indices ρ and σ. This can be done by using the *ad*-invariance of k (eq. (V.3)) to rewrite (V.5) as

$$\begin{aligned} \omega_{\rho l_2 \ldots l_{2m-2}\sigma} = &- (-1)^{\alpha(j_{2m-2})[\alpha(\rho)+\alpha(i_1)+\ldots+\alpha(i_{m-2})]} \tilde{\epsilon}^{j_2 \ldots j_{2m-2}}_{l_2 \ldots l_{2m-2}} \\ &[\sum_{s=1}^{m-2} (-1)^{\alpha(j_{2m-2})[\alpha(\rho)+\alpha(i_1)+\ldots+\alpha(i_{s-1})]} k_{\rho i_1 \ldots i_{s-1} i_{m-1} i_{s+1} \ldots i_{m-2}\sigma} C^{i_1}_{j_2 j_3} \ldots C^{i_{m-1}}_{j_{2m-2} i_s} \\ &+ k_{i_{m-1} i_1 \ldots i_{m-2}\sigma} C^{i_1}_{j_2 j_3} \ldots C^{i_{m-1}}_{j_{2m-2}\rho}] \quad . \end{aligned} \tag{V.8}$$

By using that $\alpha(i) = \alpha(j) + \alpha(k)$ if i, j, k are the indices of a C^i_{jk} commutator and the graded skew-symmetry and symmetry properties of $\tilde{\epsilon}$ and k respectively, it is easily seen that the first term is equal to

$$\begin{aligned} &\sum_{s=1}^{m-2} \tilde{\epsilon}^{j_2 \ldots j_{2s} j_{2s+1} j_{2m-2} j_{2s+2} \ldots j_{2m-3}}_{l_2 \ldots l_{2m-2}} k_{\rho i_1 \ldots i_{s-1} i_{m-1} i_{s+1} \ldots i_{m-2}\sigma} \\ &\qquad \cdot C^{i_1}_{j_2 j_3} \ldots C^{i_s}_{j_{2s} j_{2s+1}} C^{i_{m-1}}_{i_s j_{2m-2}} C^{i_{s+1}}_{j_{2s+2} j_{2s+3}} \ldots C^{i_{m-2}}_{j_{2m-4} j_{2m-3}} \quad , \end{aligned} \tag{V.9}$$

which is zero due to the ordinary super-Jacobi identity involving $C^{i_s}_{j_{2s} j_{2s-1}} C^{i_{m-1}}_{i_s j_{2m-2}}$. Thus, ω reduces to the second term in (V.8) and reads

$$\begin{aligned} \omega_{\rho l_2 \ldots l_{2m-2}\sigma} = &- (-1)^{\alpha(\rho)\alpha(\sigma)+[\alpha(\rho)+\alpha(\sigma)][\alpha(j_2)+\ldots+\alpha(j_{2m-2})]} \\ &\tilde{\epsilon}^{j_2 \ldots j_{2m-2}}_{l_2 \ldots l_{2m-2}} k_{\sigma i_1 \ldots i_{m-1}} C^{i_1}_{j_2 j_3} \ldots C^{i_{m-1}}_{j_{2m-2}\rho} \\ = &- (-1)^{\alpha(\rho)\alpha(\sigma)+[\alpha(\rho)+\alpha(\sigma)][\alpha(l_2)+\ldots+\alpha(l_{2m-2})]} \omega_{\sigma l_2 \ldots l_{2m-2}\rho} \quad , \end{aligned} \tag{V.10}$$

where the last equality is due to the fact that the presence of $\tilde{\epsilon}$ means that $\alpha(j_2) + ... + \alpha(j_{2m-2}) = \alpha(l_2) + ... + \alpha(l_{2m-2})$. Hence, ω is graded skew-symmetric.

Due to Lemma IV.1, the second part of the theorem requires checking the generalized super-Jacobi identity for $\{x_{i_1}, ..., x_{i_{2m-2}}\} = x_\sigma \omega^{\sigma}_{\cdot\; i_1 \ldots i_{2m-2}}$, which means computing

$$
\begin{aligned}
&\tilde{\epsilon}^{i_1\ldots i_{4m-5}}_{j_1\ldots j_{4m-5}} x_\sigma \omega^{\sigma}_{\cdot\ i_1\ldots i_{2m-3}\rho}\,\omega^{\rho}_{\cdot\ i_{2m-2}\ldots i_{4m-5}} \\
&= (2m-3)!\,\tilde{\epsilon}^{i_1\ldots i_{4m-5}}_{j_1\ldots j_{4m-5}} x_\sigma k^{\sigma}{}_{l_1\ldots l_{m-1}} C^{l_1}_{i_1 i_2}\ldots C^{l_{m-1}}_{i_{2m-3}\rho}\,\omega^{\rho}{}_{i_{2m-2}\ldots i_{4m-5}} \\
&= (2m-3)!(-1)^{\alpha(l_{m-1})}\tilde{\epsilon}^{i_1,\ldots,i_{4m-5}}_{j_1,\ldots,j_{4m-5}} x_\sigma k^{\sigma}{}_{l_1\ldots l_{m-2}}{}^{l_{m-1}} C^{l_1}_{i_1 i_2}\ldots C^{\rho}_{l_{m-1} i_{2m-3}}\,\omega_{\rho i_{2m-2}\ldots i_{4m-5}}\ ,
\end{aligned}
\tag{V.11}
$$

where we have used (V.5) for one of the two ω factors, that $U^i V_i = (-1)^{\alpha(i)} U_i V^i$ (which follows from the graded symmetry of the Killing matrix) and eq. (V.1). It is clear from (V.11) and (V.4) that the generalized super-Jacobi identity is satisfied if ω is ad-invariant. But this is indeed the case, due to the ad-invariance of k: substituting (V.5) in the left hand side of (V.4) and then using again (V.3),

$$
\begin{aligned}
\tilde{\epsilon}^{j_1\ldots j_{2m-1}}_{i_1\ldots i_{2m-1}} C^{k}_{\alpha j_1}\omega_{k j_2\ldots j_{2m-1}} =&(2m-3)!\,\tilde{\epsilon}^{j_1\ldots j_{2m-1}}_{i_1\ldots i_{2m-1}}\Big(\sum_{s=2}^{m}(-1)^{\alpha(j_1)[\alpha(l_2)+\ldots+\alpha(l_{s-1})]} \\
&- k_{k l_2\ldots l_{s-1} l_1 l_{s+1}\ldots l_m} C^{l_1}_{j_1 l_2} C^{l_2}_{j_2 j_3}\ldots C^{l_m}_{j_{2m-2} j_{2m-3}}\Big) = 0
\end{aligned}
\tag{V.12}
$$

which is easily seen to vanish by bringing the index j_1 next to j_{2s-1}, with the corresponding sign from $\tilde{\epsilon}$, and then using the ordinary super-Jacobi identity, *q.e.d*

In fact, it may be shown that different $\Lambda^{(2m-2)}$, $\Lambda^{(2m'-2)}$ tensors also commute with respect to the SSNB and that they are functionally independent.

In practice, given a matrix representation X_i of the superalgebra $\mathcal{G}$, the supertraces (see, *e.g.*[18]) of the graded-symmetric product of several generators define invariant polynomials *i.e.*,

$$
k_{i_1\ldots i_m} \propto \mathrm{ssTr}(X_{i_1}\ldots X_{i_m}) \quad , \tag{V.13}
$$

where ssTr means graded-symmetric supertrace and of which the Killing form k_{ij}=sTr($ad\,X_i$ $ad\,X_j$) is the lowest order example. The fact that these tensors are invariant is deduced easily from the 'cyclic' property of the supertrace, $\mathrm{sTr}(AB) = (-1)^{\alpha(A)\alpha(B)}\mathrm{sTr}(BA)$. Indeed, using the definition (V.3),

$$\sum_{s=1}^{m}(-1)^{\alpha(j)[\alpha(i_1)+...+\alpha(i_{s-1})]}C^{\rho}_{ji_s}k_{i_1...i_{s-1}\rho i_{s+1}...i_m}$$
$$= \mathrm{sTr}(\sum_{s=1}^{m}(-1)^{\alpha(j)[\alpha(i_1)+...+\alpha(i_{s-1})]}X_{i_1}...X_{i_{s-1}}[X_j, X_{i_s}]X_{i_{s+1}}...X_{i_m})$$
$$= \mathrm{sTr}([X_j, X_{i_1}]X_{i_2}...X_{i_m} + (-1)^{\alpha(j)\alpha(i_1)}X_{i_1}[X_j, X_{i_2}]X_{i_3}...X_{i_m} \tag{V.14}$$
$$+ ... + (-1)^{\alpha(j)[\alpha(i_1)+...+\alpha(i_{m-1})]}X_{i_1}...X_{i_{m-1}}[X_j, X_{i_m}])$$
$$= \mathrm{sTr}(X_jX_{i_1}...X_{i_m} - (-1)^{\alpha(j)[\alpha(i_1)+...+\alpha(i_m)]}X_{i_1}...X_{i_m}X_j) = 0 \quad .$$

VI. AN EXAMPLE OF GENERALIZED SUPER-POISSON STRUCTURE

As we have seen, the construction of a linear SPS following the procedure of Sec. V uses a graded-symmetric invariant polynomial on a simple superalgebra with non-degenerate Killing form. This does not always exist: for certain simple Lie superalgebras k is identically zero[28]. An example that does not present this problem is $su(3,1)$, the simplest superunitary algebra containing $su(3)$. A simpler example would be the unitary orthosymplectic superalgebra $uosp(2,1)$, which contains $su(2)$, but it does not have primitive graded-symmetric polynomials of order higher than two (much in the same way $su(2)$, being of rank one, has only one primitive Casimir operator) and for it eq. (V.5) reduces to the SPS given by the ω in (III.14).

Instead of making all the structure constants and commutators/anticommutators of the rank three, fifteen-generator $su(3,1)$-superalgebra explicit, we shall proceed in a more basic way which will allow us to exhibit the essentials of our general procedure. To this aim, consider first the identity

$$\epsilon^{ijkl}_{1234} = \epsilon^{ij}_{12}\epsilon^{kl}_{34} - \epsilon^{ij}_{13}\epsilon^{kl}_{24} + \epsilon^{ij}_{14}\epsilon^{kl}_{23} + \epsilon^{ij}_{23}\epsilon^{kl}_{14} - \epsilon^{ij}_{24}\epsilon^{kl}_{13} + \epsilon^{ij}_{34}\epsilon^{kl}_{12} \quad , \tag{VI.1}$$

for the standard Levi-Civita tensor. This means that the 24 terms in the four-commutator $[X_1, X_2, X_3, X_4]$, which is defined as the antisymmetrized sum of all products of the four generators, may be expressed as a sum of three terms

$$[X_1,X_2,X_3,X_4]=\{[X_1,X_2],[X_3,X_4]\}+\{[X_2,X_3],[X_1,X_4]\}+\{[X_3,X_1],[X_2,X_4]\}\quad , \tag{VI.2}$$

where $[\ ,\ ]$ ($\{\ ,\ \}$) means commutator (anticommutator). We may easily extend this to the Z_2-graded case. If we now use the Z_2-graded commutator with

$$[X,Y]:=XY-(-1)^{\alpha(X)\alpha(Y)}YX\quad (=-(-1)^{\alpha(X)\alpha(Y)}[Y,X]) \tag{VI.3}$$

(*i.e.*, the SSNB (II.14) for the elements of a superalgebra which reduces to a commutator or, in the odd/odd case, to the anticommutator), relation (VI.2) above becomes in the graded case

$$\begin{aligned}[X_i,X_j,X_k,X_l]=&\{[X_i,X_j],[X_k,X_l]\}+(-1)^{\alpha(i)\alpha(k)+\alpha(i)\alpha(j)}\{[X_j,X_k],[X_i,X_l]\}\\ &+(-1)^{\alpha(i)\alpha(k)+\alpha(j)\alpha(k)}\{[X_k,X_i],[X_j,X_l]\}\quad ,\end{aligned} \tag{VI.4}$$

where $\{\ ,\ \}$ is now the Z_2-graded *anticommutator* defined by

$$\{X,Y\}:=XY+(-1)^{\alpha(X)\alpha(Y)}YX\quad (=(-1)^{\alpha(X)\alpha(Y)}\{Y,X\})\quad . \tag{VI.5}$$

Expression (VI.4) of course follows from the equivalent to (VI.1) for the Z_2-graded case *i.e.*,

$$\begin{aligned}\tilde{\epsilon}^{ijkl}_{1234}=&\tilde{\epsilon}^{ij}_{12}\tilde{\epsilon}^{kl}_{34}-(-1)^{\alpha(2)\alpha(3)}\tilde{\epsilon}^{ij}_{13}\tilde{\epsilon}^{kl}_{24}\\ &+(-1)^{\alpha(4)(\alpha(2)+\alpha(3))}\tilde{\epsilon}^{ij}_{14}\tilde{\epsilon}^{kl}_{23}+(-1)^{\alpha(1)(\alpha(2)+\alpha(3))}\tilde{\epsilon}^{ij}_{23}\tilde{\epsilon}^{kl}_{14}\\ &-(-1)^{\alpha(1)\alpha(2)+\alpha(4)(\alpha(1)+\alpha(3))}\tilde{\epsilon}^{ij}_{24}\tilde{\epsilon}^{kl}_{13}+(-1)^{(\alpha(3)+\alpha(4))(\alpha(1)+\alpha(2))}\tilde{\epsilon}^{ij}_{34}\tilde{\epsilon}^{kl}_{12}\quad .\end{aligned} \tag{VI.6}$$

Let us write the graded commutators among the elements of a basis of $su(3,1)$ as $[X_a,X_b]=C^c_{ab}X_c$. For the graded anticommutators (VI.5) we have, in *this* case, the generic form

$$\{X_a,X_b\}=k^c_{ab}X_c+\delta_{ab}I\quad , \tag{VI.7}$$

where I represents a central element. This means that the *r.h.s* of (VI.4) above may be written as

$$\begin{aligned}&C^m_{ij}C^n_{kl}(k^p_{mn}X_p+\delta_{mn})+(-1)^{\alpha(i)\alpha(k)+\alpha(i)\alpha(j)}C^m_{jk}C^n_{il}(k^p_{mn}X_p+\delta_{mn})\\ &\qquad\qquad+(-1)^{\alpha(i)\alpha(k)+\alpha(j)\alpha(k)}C^m_{ki}C^n_{jl}(k^p_{mn}X_p+\delta_{mn})\quad .\end{aligned} \tag{VI.8}$$

It is easy to see that terms with δ cancel by using the graded Jacobi identity (III.15) for the structure constants. Grouping the terms in k, we may see that

$$[X_i, X_j, X_k, X_l] = \frac{1}{2}\tilde{\epsilon}^{rst}_{ijk} C^m_{rs} C^n_{tl} k^p_{mn} X_p = \frac{1}{2}\omega^p_{ijkl} X_p \quad , \tag{VI.9}$$

where the ω appearing here has, precisely, the structure of (V.5). Using this result, it is clear that $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\} = x_\sigma \omega^\sigma_{i_1 i_2 i_3 i_4}$ (see (V.7)) defines a linear GSPS on $su(3,1)^*$. Note that this depends on the existence of the graded symmetric polynomial k^c_{ab} in (VI.7), which is a specific property of $su(3,1)$. However, the procedure may be extended to other algebras along similar lines.

ACKNOWLEDGEMENTS

This research has been partially supported by the CICYT and the DGICYT, Spain (AEN 96-1669, PR 95-439). J.M.I thanks the HCM programme of the European Union for financial support; A.P. thanks the Vice-rectorate of research of Valencia University for making his stay in Valencia possible. J.A. and J.C.P.B. wish to acknowledge the kind hospitality extended to them at DAMTP; they also wish to thank St. John's College for support (J.A.) and the Spanish Ministry of Education and Science and the CSIC for an FPI grant (J.C.P.B.).

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